

EVALUATION OF STATE INTEGRALS AT RATIONAL POINTS

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ABSTRACT. Multi-dimensional state-integrals of products of Faddeev's quantum dilogarithms arise frequently in Quantum Topology, quantum Teichmüller theory and complex Chern–Simons theory. Using the quasi-periodicity property of the quantum dilogarithm, we evaluate 1-dimensional state-integrals at rational points and express the answer in terms of the Rogers dilogarithm, the cyclic (quantum) dilogarithm and finite state-sums at roots of unity. We illustrate our results with the evaluation of the state-integrals of the 4_1 , 5_2 and $(-2, 3, 7)$ pretzel knots at rational points.

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1. INTRODUCTION

1.1. State-integrals and their q -series. State-integrals are multi-dimensional integrals of products of Faddeev's quantum dilogarithms. They appear in abundance in Quantum Topology, quantum Teichmüller theory and in complex Chern–Simons theory. State integrals were studied among others by Hikami [Hik01], Dimofte–Gukov–Lennels–Zagier [DGLZ09],

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Andersen–Kashaev [AK14, AK], Kashaev–Luo–Vartanov [KLV12], Dimofte [Dim14a] and Dimofte–Garoufalidis [Dim14b].

In our previous paper [GK], we showed how to express 1-dimensional state-integrals as a finite sum of products of q -series and \tilde{q} -series with integer coefficients, where the variables q and \tilde{q} are related by the modular transformation: $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$.

In this paper we evaluate 1-dimensional state-integrals at rational points in terms of the Rogers dilogarithm, the cyclic (quantum) dilogarithm and truncated state-sums at roots of unity. Our formulas are syntactically similar with

- (a) the constant terms of the power series that appear in the Quantum Modularity Conjecture of Zagier [Zag10, GZa],
- (b) the 1-loop terms of the perturbation expansion of complex Chern–Simons theory [Dim14b],
- (c) the state-sums of quantum Teichmüller theory [Kas94, Kas95, Kas97] and also [BB07, Sec.6].

This is not a coincidence; it is one part of a story discussed in detail in [GZb].

In order to keep our principle clear, we focus exclusively on 1-dimensional state-integrals, and we illustrate our results for the state-integrals of 4_1 , 5_2 and $(-2, 3, 7)$ pretzel knots. In a separate publication we will discuss the evaluation of multi-dimensional state-integrals.

A 1-dimensional state-integral is an absolutely convergent integral of the form

$$(1) \quad \mathcal{I}_{A,B}(\mathbf{b}) = \int_{\mathbb{R}+i\epsilon} \Phi_{\mathbf{b}}(x)^B e^{-A\pi i x^2} dx$$

for a complex number \mathbf{b} with $\mathbf{b}^2 \notin \mathbb{R}_{\leq 0}$. Here A, B are natural numbers satisfying $B > A > 0$ and $\Phi_{\mathbf{b}}(x)$ is Faddeev’s quantum dilogarithm function [Fad95, FK94]. Few properties of this special function are reviewed in Appendix A.

A numerical computation by the first author and Zagier [GZb] suggested the following formula for $\mathcal{I}_{1,2}(1)$:

$$(2) \quad \mathcal{I}_{1,2}(1) = \frac{e^{\pi i/6}}{\sqrt{3}} \left(e^{\frac{V}{2\pi}} - e^{-\frac{V}{2\pi}} \right)$$

(and more generally for the Taylor coefficients of the analytic function $\mathcal{I}_{1,2}(\mathbf{b})$ at $\mathbf{b} = 1$), where $V = 2 \operatorname{Im}(\operatorname{Li}_2(e^{\pi i/3})) = 2.0298832 \dots$ is the volume of the 4_1 knot. Understanding and proving the above identity led to the results of our paper.

Our aim is to evaluate $\mathcal{I}_{A,B}(\mathbf{b})$ when $\mathbf{b}^2 = M/N$ for a pair of coprime natural numbers M, N . The content of our paper can be summarized in a diagram

$$\begin{array}{ccc} \{\text{state-integrals}\} & \longrightarrow & \{\text{Nahm series}\} \\ \downarrow & & \downarrow \\ \{\text{evaluations}\} & \longleftarrow & \{\text{truncated Nahm series}\} \end{array}$$

The top arrow was the content of our previous article [GK]. To recall the connection between state-integrals and q -series, consider the integrand of the state-integral $\mathcal{I}_{A,B}(\mathbf{b})$, shifted by

$c_{\mathbf{b}} = i(\mathbf{b} + \mathbf{b}^{-1})/2$:

$$(3) \quad f(x - c_{\mathbf{b}}) = \Phi_{\mathbf{b}}(x)^B e^{-A\pi i x^2}.$$

The quasi-periodicity of the quantum dilogarithm (see Equations (77a)–(77b)) implies that

$$f(x + im\mathbf{b} + in\mathbf{b}^{-1}) = f(x) g_m^+(e^{2\pi i \mathbf{b} x}, q_+) g_n^-(e^{2\pi i \mathbf{b}^{-1} x}, q_-)$$

where $q_{\pm} = e^{2\pi i \mathbf{b}^{\pm 2}}$ and

$$(4) \quad g_k^{\pm}(x, q) = (-x)^{Ak} \frac{q^{\frac{A}{2}k(k+1)}}{(qx; q)_k^B}.$$

This gives rise to the series $G^{\pm}(x, q) \in \mathbb{Z}[[x, q]]$ defined by

$$(5) \quad G^{\pm}(x, q) = \sum_{k=0}^{\infty} g_k^{\pm}(x, q)$$

The q -series $G^{\pm}(1, q) \in \mathbb{Z}[[q]]$ are special q -hypergeometric series of Nahm type and appear in the expression of the state-integral $\mathcal{I}_{A,B}(b)$ as a sum of products of q -series and \tilde{q} -series, where $\tilde{q} = 1/q_-$, see [GK, Thm.1.1].

Throughout the paper, (M, N) will denote an *admissible* pair, i.e., a pair of coprime positive integers. Consider the state-sum defined by

$$(6) \quad G_{M,N}(x_+, x_-) = \sum_{k=0}^{MN-1} g_{kP}^+(x_+, \zeta_N^M) g_{kQ}^-(x_-, \zeta_M^N)$$

where P, Q are integers that satisfy the equation $MP + NQ = 1$ and $\zeta_N = e^{2\pi i/N}$. When $x_+^N = x_-^M$, it follows from Lemma 2.2 that $G_{M,N}(x_+, x_-)$ is independent of the choice of P and Q . Observe that

$$G_{1,N}(x_+, x_-) = G_N^+(x_+), \quad G_{M,1}(x_+, x_-) = G_M^-(x_-)$$

where

$$(7) \quad G_N^{\pm}(x) = \sum_{k=0}^{N-1} g_k^{\pm}(x, \zeta_N).$$

1.2. The Rogers and the cyclic dilogarithms. Recall the *Rogers dilogarithm* [Neu04, GZ07]

$$(8) \quad R(z) = \text{Li}_2(z) + \frac{1}{2} \log(z) \log(1-z) - \frac{\pi^2}{6}$$

and its extension as a multivalued function on the universal abelian cover of $\mathbb{C} \setminus \{0, 1\}$.

The *cyclic (quantum) dilogarithm* $D_N(x; q)$ is the N -th root of a polynomial in x with constant term 1 defined by

$$(9) \quad D_N(x; q) = \prod_{k=1}^{N-1} (1 - q^k x)^{k/N}.$$

It appeared in [KMS93, Eqn.C.3] and [Kas99, Eqn.2.30], and its N -th power is characterized among polynomials by the functional equation

$$\frac{D_N(\zeta_N x; \zeta_N)^N}{D_N(x; \zeta_N)^N} = \frac{(1-x)^N}{1-x^N}, \quad D_N(0)^N = 1.$$

It will be useful to introduce the following variant \mathcal{D}_N defined by

$$(10) \quad \mathcal{D}_N(x; q) = \prod_{k=1}^N (1 - xq^k)^{k/N} = (1 - xq^N) D_N(x; q).$$

1.3. Evaluation of state-integrals. Our main theorem evaluates the state-integral at $\mathbf{b}^2 = M/N$ in terms of the state-sums $G_{M,N}$, the Rogers dilogarithm and the cyclic dilogarithm.

Fix an admissible pair (M, N) , and define

$$(11) \quad \mathbf{b} = \sqrt{M/N}, \quad \mathbf{s} = \sqrt{MN}.$$

Let

$$(12) \quad \mathcal{S} = \{w \mid g(e^{2\pi s w}) = 1, \ 0 < \mathbf{s} \operatorname{Im}(w) - \lambda < 1\},$$

where λ is a generic real number such that

$$(13) \quad -(M+N)/2 < \lambda < 0$$

and

$$(14) \quad g(z) = (-z)^A (1-z)^{-B} \in \mathbb{Q}[z^{\pm 1}].$$

Note that if $w \in \mathcal{S}$, then $e^{2\pi s w}$ is an algebraic number with a fixed choice of N and M -th roots.

Theorem 1.1. *When $\mathbf{b}^2 = M/N$ we have:*

$$(15) \quad \mathcal{I}_{A,B}(\mathbf{b}) = e^{\pi i \frac{B+3A(M+N+1)^2-6MN}{12MN}} \mathbf{s}^{-1} \sum_{w \in \mathcal{S}} \frac{e^{\frac{iB}{2\pi s^2} \operatorname{R}(z)} (1-z)^{\frac{(2N+1)(2M+1)}{4MN} B}}{z g'(z) \mathcal{D}_N(\theta_+, q_+)^B \mathcal{D}_M(\theta_-, q_-)^B} G_{M,N}(\theta_+, \theta_-),$$

where

$$(16) \quad z = e^{2\pi s w}, \quad \theta_+ = e^{2\pi \mathbf{b} w} = z^{1/N}, \quad \theta_- = e^{2\pi \mathbf{b}^{-1} w} = z^{1/M}, \quad q_+ = \zeta_N^M, \quad q_- = \zeta_M^N.$$

Note that when $g(z) = 1$, we have

$$(17) \quad g'(z) = Az^{-1} + B(1-z)^{-1}.$$

Corollary 1.2. For $M = 1$ we obtain that

$$(18) \quad \mathcal{I}_{A,B}(\mathbf{b}) = e^{\pi i \frac{B+3A(N+2)^2-6N}{12N}} \frac{1}{\sqrt{N}} \sum_{w \in \mathcal{S}} e^{\frac{iB}{2\pi N} \operatorname{R}(z)} \frac{(1-z)^{\frac{2N+3}{4N} B}}{(A + Bz/(1-z)) \mathcal{D}_N(\theta_+, q_+)^B} G_N^+(\theta_+).$$

When $M = N = 1$ we obtain that

$$(19) \quad \mathcal{I}_{A,B}(1) = e^{\pi i \frac{B+3A-6}{12}} \sum_{w \in \mathcal{S}} e^{\frac{iB}{2\pi} \operatorname{R}(z)} \frac{(1-z)^{\frac{B}{4}}}{(A + Bz/(1-z))}.$$

Let us denote

$$(20) \quad \mathbf{e}(x) = e^{2\pi i x}.$$

Corollary 1.3. When $M = N = 1$ and $(A, B) = (1, 2)$, we choose λ to be a negative real number near zero,

$$\begin{aligned} g(z) &= -z(1-z)^{-2} \\ \mathcal{S} &= \{i/6, 5i/6\} \\ z_{\pm} &= \mathbf{e}(\pm 1/6) \\ e^{\pi i \frac{B+3A-6}{12}} &= \mathbf{e}\left(-\frac{1}{24}\right) \\ (e^{\frac{iB}{2\pi} R(z_+)}, e^{\frac{iB}{2\pi} R(z_-)}) &= \left(e^{-C} \mathbf{e}\left(-\frac{1}{24}\right), -e^C \mathbf{e}\left(-\frac{1}{24}\right) \mathbf{e}\left(\frac{1}{3}\right)\right) \\ \frac{(1-z_{\pm})^{\frac{B}{4}}}{(A+Bz_{\pm}/(1-z_{\pm}))} &= \frac{1}{\sqrt{3}} \mathbf{e}\left(\mp \frac{1}{3}\right) \end{aligned}$$

where $C = V/(2\pi)$ and V is the volume of the 4_1 knot. When computing the Rogers dilogarithm of z_{\pm} , keep in mind that we use the branches of the logarithm $\log z_+ = 2\pi i/6$ and $\log z_- = 10\pi i/6$ dictated by Equations (12) and (13).

The above computation, combined with Equation (19) implies Equation (2). As was already mentioned, the proof of this equation was a main motivation for the results of our paper.

We now make few remarks about the number-theoretic, analytic and geometric properties of Equation (15).

Remark 1.4. It [GZa] (see also [Dim14b]) it was observed that although $(G^+(\theta_+))^N$ and $(\mathcal{D}_N(\theta_+, \zeta_N))^N$ lie in the field $F_{G,N} = \mathbb{Q}(\theta_+, \zeta_N)$, their ratio lies in the smaller field $\mathbb{Q}(z, \zeta_N)$ (where z satisfies $g(z) = 1$) which is an extension of $\mathbb{Q}(z)$ by ζ_N . In particular, the above mentioned ratio is independent of the choice of the N -th root of z .

Remark 1.5. Although \mathcal{D}_N is a multivalued function, the sum in Equation (15) is well-defined. This is a consequence of Theorem 1.9 below and the fact that the quantum dilogarithm is a meromorphic function.

Remark 1.6. When the state-integral is associated with a cusped hyperbolic manifold M , the set \mathcal{S} is often in bijection with the set of nonabelian parabolic $\mathrm{PSL}(2, \mathbb{C})$ representations of M . Under such a bijection, the Rogers dilogarithm matches with the complex volume, and the value of $g'(z)$ matches with the value of the 1-loop invariant of [DG13], suitably normalized. For an illustration, see Section 2.3.

Remark 1.7. When the state-integral is associated with a cusped hyperbolic manifold M and the identification of Remark 1.6 is available, one can identify Equation (15) with a sum of invariants of M parametrized by nonabelian parabolic $\mathrm{PSL}(2, \mathbb{C})$ representations of M . Such invariants appear in Quantum Hyperbolic Geometry—see [Kas97] and also [BB07]. The invariants of Quantum Hyperbolic Geometry are defined up to multiplication by an N -root

of unity. However, Equation (15) gives a well-defined *relative* choice of the N -th roots of unity. This is a consequence of the meromorphicity of the quantum dilogarithm.

Remark 1.8. As we already mentioned above, a numerical computation by the first author and Zagier suggests an explicit formula for the Taylor series of $\mathcal{I}_{1,2}(\mathbf{b})$ at $\mathbf{b} = 1$ in terms of the asymptotics of the Kashaev invariant at $q = 1$. We expect that the Taylor series of state-integrals at $\mathbf{b} = \sqrt{M/N}$ can be expressed in terms of the loop invariants of Garoufalidis–Dimofte [Dim14b]. We plan to study this in a later publication.

Theorem 1.1 follows from a lemma from complex analysis regarding integrals of quasi-periodic functions 2.3. This lemma is used twice, once to evaluate the quantum dilogarithm in terms of the cyclic dilogarithm, and another time to evaluate the state-integral $\mathcal{I}_{A,B}(\mathbf{b})$.

1.4. The quantum dilogarithm at roots of unity. We fix an admissible pair (M, N) . Recall \mathbf{b} and \mathbf{s} from Equation (11). Let $\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ denote the Euler dilogarithm, defined for $|z| < 1$ and analytically continued as a multivalued function on $\mathbb{C} \setminus \{0, 1\}$.

Theorem 1.9. *We have:*

$$(21) \quad \Phi_{\mathbf{b}}\left(\frac{z}{2\pi\mathbf{s}} - c_{\mathbf{b}}\right) = \frac{e^{\frac{i}{2\pi\mathbf{s}^2} \text{Li}_2(e^z)} (1 - e^z)^{1 + \frac{iz}{2\pi\mathbf{s}^2}}}{D_N(e^{z/N}; q_+) D_M(e^{z/M}; q_-)}.$$

It is remarkable that the left-hand side is a meromorphic function of z whereas the right hand side is assembled out of multivalued functions of z .

In particular when $M = 1$, we obtain that

$$(22) \quad \Phi_{\mathbf{b}}(x - c_{\mathbf{b}}) = \frac{e^{-\frac{1}{2\pi i N} \text{Li}_2(z^N)}}{D_N(z)} (1 - z^N)^{1 + \frac{ix}{\sqrt{N}}}, \quad z = e^{2\pi\mathbf{b}x},$$

and when $M = N = 1$, we obtain that

$$(23) \quad \Phi_1(x) = \exp\left(\frac{i}{2\pi} (\text{Li}_2(e^{2\pi x}) + 2\pi x \log(1 - e^{2\pi x}))\right).$$

By using the equality

$$(24) \quad \frac{\Phi_{\mathbf{b}}\left(\frac{z}{2\pi\mathbf{s}} - c_{\mathbf{b}}\right)}{\Phi_{\mathbf{b}}\left(\frac{z}{2\pi\mathbf{s}} + c_{\mathbf{b}}\right)} = (1 - e^{z/N}) (1 - e^{z/M})$$

we also have

$$(25) \quad \Phi_{\mathbf{b}}\left(\frac{z}{2\pi\mathbf{s}} + c_{\mathbf{b}}\right) = \frac{e^{\frac{i}{2\pi\mathbf{s}^2} \text{Li}_2(e^z)} (1 - e^z)^{1 + \frac{iz}{2\pi\mathbf{s}^2}}}{\bar{D}_N(e^{z/N}; q_+) \bar{D}_M(e^{z/M}; q_-)},$$

Remark 1.10. The cyclic dilogarithm is in a sense a radial limit of the generating series

$$M(x, q) = \prod_{n=1}^{\infty} \frac{1}{(1 - xq^n)^n}$$

where $M(x, q)$ is the McMahon generating series of 3-dimensional *plane partitions*; see [AKMV05] and [ORV06, Sec.2.1]. The latter appear in M-theory and mirror symmetry. It would be interesting and useful to understand a precise relation between the function D_N and plane partitions.

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2. INTEGRALS OF QUASI-PERIODIC FUNCTIONS

2.1. Some lemmas from complex analysis.

Lemma 2.1. Let $f: \mathcal{U} \rightarrow \mathbb{C}$ be an analytic function satisfying the functional equation

$$(26) \quad f(z-a)f(z+a) = f(z)^2,$$

with some fixed $a \in \mathbb{C} \setminus \{0\}$, the domain $\mathcal{U} \subset \mathbb{C}$ being a translationally invariant open set, $\mathcal{U} = a + \mathcal{U}$, and $\mathcal{C} \subset \mathcal{U}$ an oriented path such that $f(z)(f(z) - f(z+a)) \neq 0$ for all $z \in \mathcal{C}$. Then

$$(27) \quad \int_{\mathcal{C}} f(z) dz = \left(\int_{\mathcal{C}} - \int_{a+\mathcal{C}} \right) \frac{f(z)}{1 - f(z+a)/f(z)} dz.$$

Proof. We have

$$(28) \quad \begin{aligned} \left(\int_{\mathcal{C}} - \int_{a+\mathcal{C}} \right) \frac{f(z)}{1 - f(z+a)/f(z)} dz &= \int_{\mathcal{C}} \frac{f(z)}{1 - f(z+a)/f(z)} dz \\ &\quad - \int_{a+\mathcal{C}} \frac{f(z)}{1 - f(z)/f(z-a)} dz = \int_{\mathcal{C}} \frac{f(z) - f(z+a)}{1 - f(z+a)/f(z)} dz = \int_{\mathcal{C}} f(z) dz. \end{aligned}$$

□

Given a rational function $r(z) \in \mathbb{C}(z)$, a nonzero complex number $q \in \mathbb{C} \setminus \{0\}$ and an integer k we define $r_k(z; q) \in \mathbb{C}(z)$, $k \in \mathbb{Z}$, by:

$$(29) \quad \frac{r_{k+1}(z; q)}{r_k(z; q)} = r(zq^k), \quad r_0(z; q) = 1.$$

Note that

$$(30) \quad r_1(z; q) = r(z)$$

and

$$(31) \quad r_{k+l}(z; q) = r_k(zq^l; q) r_l(z; q) = r_l(zq^k; q) r_k(z; q),$$

for all integers k, l . In particular, in the case of roots of unity this implies a quasi-periodicity property:

$$(32) \quad r_{k+N}(z; q) = r_k(z; q) r_N(z; q), \quad \text{if } q^N = 1,$$

and the invariance property

$$(33) \quad r_N(zq; q) = r_N(z; q), \quad \text{if } q^N = 1.$$

Let (M, N) be an admissible pair, and recall \mathbf{b} and \mathbf{s} from Equation (11). Choose two integers P and Q which satisfy the equation $MP + NQ = 1$. Let $f(z)$ be a meromorphic function and $g^\pm(z) \in \mathbb{C}(z)$ two rational functions such that

$$(34) \quad \frac{f(z + \mathbf{b}^{\pm 1} \mathbf{i})}{f(z)} = g^\pm(e^{2\pi \mathbf{b}^{\pm 1} z}).$$

Then, for $k \in \mathbb{Z}$ we have

$$(35) \quad g^\pm_k(e^{2\pi \mathbf{b}^{\pm 1} z}; q_\pm) = \frac{f(z + \mathbf{b}^{\pm 1} \mathbf{i} k)}{f(z)},$$

and, in particular,

$$(36) \quad g^+_N(e^{2\pi \mathbf{b} z}; q_+) = \frac{f(z + \mathbf{b} \mathbf{i} N)}{f(z)} = \frac{f(z + \mathbf{s} \mathbf{i})}{f(z)} = \frac{f(z + \mathbf{b}^{-1} \mathbf{i} M)}{f(z)} = g^-_M(e^{2\pi \mathbf{b}^{-1} z}; q_-).$$

Define

$$(37) \quad g(x) = g^+_N(x^{\frac{1}{N}}; q_+) = g^-_M(x^{\frac{1}{M}}; q_-)$$

$$(38) \quad S(x, y) = \sum_{k=0}^{s^2-1} g^+_{kP}(x; q_+) g^-_{kQ}(y; q_-)$$

Lemma 2.2. (a) We have $g(x) \in \mathbb{C}(x)$ and $S(x, y) \in \mathbb{C}(x, y)$.

(b) The function $S(x, y)$ is independent of P and Q provided $x^N = y^M$.

Proof. Since

$$(39) \quad g^+_N(xq_+; q_+) = g^+_N(x; q_+), \quad g^-_M(xq_-; q_-) = g^-_M(x; q_-),$$

it follows that $g(x) \in \mathbb{C}(x)$, and consequently, $S(x, y) \in \mathbb{C}(x, y)$.

For (b), let P', Q' be another pair satisfying the equation $MP' + NQ' = 1$. Then there exists an integer R such that $P' = P + RN$ and $Q' = Q - RM$. Denoting the function (38) with P and Q replaced by P' and Q' as $S'(x, y)$, we have

$$\begin{aligned} S'(x, y) &= \sum_{k=0}^{s^2-1} g^+_{kP'}(x; q_+) g^-_{kQ'}(y; q_-) = \sum_{k=0}^{s^2-1} g^+_{kP+kRN}(x; q_+) g^-_{kQ-kRM}(y; q_-) \\ &= \sum_{k=0}^{s^2-1} g^+_{kP}(x; q_+) g^+_N(x; q_+)^{kR} g^-_{kQ}(y; q_-) g^-_M(y; q_-)^{-kR} \\ &= \sum_{k=0}^{s^2-1} g^+_{kP}(x; q_+) \left(\frac{g^+_N(x; q_+)}{g^-_M(y; q_-)} \right)^{kR} g^-_{kQ}(y; q_-) = \sum_{k=0}^{s^2-1} g^+_{kP}(x; q_+) g^-_{kQ}(y; q_-) \\ &= S(x, y) \end{aligned}$$

where we used the second equality in (37). □

For a complex number x , we denote $\mathcal{C}_x = xi/s + \mathbb{R} \subset \mathbb{C}$.

Lemma 2.3. Let $f(z)$, $g(z)$, $S(x, y)$ be as above and λ a real number in general position such that the form $f(z) dz$ is absolutely integrable along \mathcal{C}_λ ,

$$(40) \quad \lim_{x \rightarrow \pm\infty} \sup_{y \in [\frac{\lambda}{s}, \frac{\lambda}{s} + s]} |f(x + iy)| = 0,$$

and

$$(41) \quad g(0) \neq 1 \neq g(\infty).$$

Then the following equalities hold

$$(42) \quad \begin{aligned} \int_{\mathcal{C}_\lambda} f(z) dz &= \left(\int_{\mathcal{C}_\lambda} - \int_{\mathcal{C}_{\lambda+1}} \right) \frac{f(z) S(e^{2\pi b z}, e^{2\pi b^{-1} z})}{1 - g(e^{2\pi s z})} dz \\ &= 2\pi i \sum_{0 < s \operatorname{Im} \alpha - \lambda < 1} \operatorname{Res}_{z=\alpha} \frac{f(z) S(e^{2\pi b z}, e^{2\pi b^{-1} z})}{1 - g(e^{2\pi s z})}. \end{aligned}$$

Proof. Let us derive the first equality. We denote

$$(43) \quad q_\pm = e^{2\pi i b^{\pm 2}},$$

so that we have

$$(44) \quad q_+ = e^{2\pi i \frac{M}{N}}, \quad q_- = e^{2\pi i \frac{N}{M}}.$$

We also have

$$(45) \quad f(z + k b^{\pm 1} i) = f(z) g_\pm^k(e^{2\pi b^{\pm 1} z}; q_\pm), \quad \forall k \in \mathbb{Z}.$$

In particular,

$$(46) \quad f(z + si) = f(z + Nbi) = f(z + Mb^{-1}i) = f(z) g(e^{2\pi s z}).$$

The function

$$(47) \quad h(z) = f\left(\frac{z}{2\pi s}\right)$$

has the properties

$$(48) \quad \frac{h(z + k2\pi i)}{h(z)} = g^+_{Pk}(e^{z/N}; q_+) g^-_{Qk}(e^{z/M}; q_-), \quad \forall k \in \mathbb{Z}.$$

From equation (48), it follows that

$$(49) \quad \sum_{k=0}^{s^2-1} \frac{h(z + k2\pi i)}{h(z)} = S(e^{z/N}, e^{z/M}).$$

As λ is generic, the contour \mathcal{C}_λ satisfies the conditions of Lemma 2.1 with $a = \text{si}$. Thus, we write

$$\begin{aligned}
 (50) \quad \int_{\mathcal{C}_\lambda} f(z) dz &= \left(\int_{\mathcal{C}_\lambda} - \int_{\text{si} + \mathcal{C}_\lambda} \right) \frac{f(z)}{1 - g(e^{2\pi s z})} dz \\
 &= \left(\int_{\mathcal{C}_{\lambda 2\pi s}} - \int_{\mathcal{C}_{(\lambda + s^2) 2\pi s}} \right) \frac{h(z)}{1 - g(e^z)} \frac{dz}{2\pi s} \\
 &= \sum_{k=0}^{s^2-1} \left(\int_{\mathcal{C}_{(\lambda+k) 2\pi s}} - \int_{\mathcal{C}_{(\lambda+k+1) 2\pi s}} \right) \frac{h(z)}{1 - g(e^z)} \frac{dz}{2\pi s} \\
 &= \sum_{k=0}^{s^2-1} \left(\int_{\mathcal{C}_{\lambda 2\pi s}} - \int_{\mathcal{C}_{(\lambda+1) 2\pi s}} \right) \frac{h(z + k 2\pi i)}{1 - g(e^z)} \frac{dz}{2\pi s} \\
 &= \left(\int_{\mathcal{C}_{\lambda 2\pi s}} - \int_{\mathcal{C}_{(\lambda+1) 2\pi s}} \right) \frac{h(z) S(e^{z/N}, e^{z/M})}{1 - g(e^z)} \frac{dz}{2\pi s} \\
 &= \left(\int_{\mathcal{C}_\lambda} - \int_{\mathcal{C}_{\lambda+1}} \right) \frac{f(z) S(e^{2\pi b z}, e^{2\pi b^{-1} z})}{1 - g(e^{2\pi s z})} dz.
 \end{aligned}$$

The second equality in Equation (42) follows from the residue theorem. To justify its application note that $\mathcal{C}_\lambda - \mathcal{C}_{\lambda+1}$ union two vertical segments is the boundary of a disk. Conditions (40), (41) imply that for sufficiently large negative or positive x the integrand in last part of (50) is regular on the vertical segment $\mathcal{C}' = x + i[\lambda, \lambda + 1]/s$ and the following estimate holds

$$\begin{aligned}
 \left| \int_{\mathcal{C}'} \frac{f(z) S(e^{2\pi b z}, e^{2\pi b^{-1} z})}{1 - g(e^{2\pi s z})} dz \right| &\leq \sum_{k=0}^{s^2-1} \int_{\lambda}^{\lambda+1} \left| \frac{f(x + i \frac{y+k}{s})}{1 - g(e^{2\pi(sx+iy)})} \right| \frac{dy}{s} \\
 &\leq \frac{1}{s} \sum_{k=0}^{s^2-1} \sup_{y \in [\lambda, \lambda+1]} \left| \frac{f(x + i \frac{y+k}{s})}{1 - g(e^{2\pi(sx+iy)})} \right| \\
 &\leq \frac{1}{s} \sup_{y \in [\lambda, \lambda+1]} \frac{1}{|1 - g(e^{2\pi(sx+iy)})|} \sum_{k=0}^{s^2-1} \sup_{t \in [\lambda, \lambda+1]} \left| f\left(x + i \frac{t+k}{s}\right) \right| \\
 &\leq s \frac{\sup_{z \in x + i[\frac{\lambda}{s}, \frac{\lambda}{s} + s]} |f(z)|}{\inf_{|w|=e^{2\pi s x}} |1 - g(w)|} \xrightarrow{|x| \rightarrow \infty} 0
 \end{aligned}$$

□

2.2. Applications to 1-dimensional state-integrals: proof of Theorem 1.1. In this section we prove Theorem 1.1. Fix integers A and B with $B > A > 0$. The values of particular interest are $(A, B) = (1, 2)$ and $(1, 3)$ which correspond to the state-integrals of the knots 4_1 and 5_2 respectively. For the 4_1 knot, see [AK14, Eqn.38] and [KLV12, Eqn.47] and [GK]. For the 5_2 knot, see [KLV12, Eqn.53]. For the remaining values of (A, B) , although

the 1-dimensional state integral makes sense, and it can be analyzed using our methods, there is no corresponding knot that we know of.

If

$$(51) \quad f(z - c_b) = \Phi_b(z)^B e^{-A\pi iz^2}, \quad c_b = (b + b^{-1})i/2,$$

then

$$(52a) \quad g(x) = (-x)^A (1-x)^{-B}$$

$$(52b) \quad g^\pm(x) = g(q_\pm x)$$

$$(52c) \quad g^\pm_n(x; q_\pm) = (-x)^{An} q_\pm^{\frac{A}{2}n(n+1)} (q_\pm x; q_\pm)_n^{-B}, \quad \forall n \in \mathbb{Z}.$$

Observe that $f(z)$ is non-vanishing and absolutely integrable along the line \mathcal{C}_λ if

$$(53) \quad -(M+N)/2 < \lambda < 0,$$

and $f(z + si) \neq f(z)$ if λ is in general position. By using (25), we obtain that

$$(54) \quad \begin{aligned} f(z) &= \Phi_b(z + c_b)^B e^{-A\pi i(z+c_b)^2} \\ &= \frac{e^{\frac{iB}{2\pi s^2} \text{Li}_2(e^{2\pi sz})} (1 - e^{2\pi sz})^{B+Bis s^{-1}}}{\mathcal{D}_N(e^{2\pi bz}; q_+)^B \mathcal{D}_M(e^{2\pi b^{-1}z}; q_-)^B} e^{-A\pi i(z+c_b)^2}. \end{aligned}$$

By using the identity

$$\begin{aligned} e^{-A\pi i(z+c_b)^2} &= e^{-A\pi i(c_b + \frac{i}{2s})^2} e^{A(2\pi sz - \pi i)(1-4sib - 2siz)s^{-2}/4} \\ &= e^{A\pi i \frac{(M+N+1)^2}{4MN}} e^{A(2\pi sz - \pi i)(1-4sib - 2siz)s^{-2}/4} \end{aligned}$$

and (52a), we can rewrite (54) in the form

$$(55) \quad f(z) = \frac{e^{\frac{iB}{2\pi s^2} R(e^{2\pi sz})(1-e^{2\pi sz})^{\frac{(2N+1)(2M+1)}{4MN}B}}}{\mathcal{D}_N(e^{2\pi bz}; q_+)^B \mathcal{D}_M(e^{2\pi b^{-1}z}; q_-)^B} e^{\pi i \frac{B+3A(M+N+1)^2}{12MN}} g(e^{2\pi sz})^{\frac{1-4sib - 2siz}{4s^2}},$$

where $R(x)$ is the Rogers dilogarithm (8).

It is easy to see that the only singularities in Equation (42) are simple poles that come from solutions to the equation $1 - g(e^{2\pi sz}) = 1$. Moreover, if $z = \alpha$ is a solution with $0 < s \text{Im} \alpha - \lambda < 1$, then

$$2\pi i \text{Res}_{z=\alpha} \frac{f(z) S(e^{2\pi bz}, e^{2\pi b^{-1}z})}{1 - g(e^{2\pi sz})} = i^{-1} s^{-1} \frac{f(\alpha) S(e^{2\pi b\alpha}, e^{2\pi b^{-1}\alpha})}{e^{2\pi s\alpha} g'(e^{2\pi s\alpha})}.$$

Combining Lemma 2.3 with Equation (55) concludes the proof of Theorem 1.1. \square

2.3. The case of the $(-2, 3, 7)$ pretzel knot. The second author computed the state-integral invariant of the $(-2, 3, 7)$ pretzel knot. The result of the long computation is the following 1-dimensional state integral, which we take as input to our analysis:

$$(56) \quad I_{(-2,3,7)}(b) = \int_{\mathbb{R}+i\epsilon} \Phi_b(x)^2 \Phi_b(2x - c_b) e^{-2\pi i x^2} dx$$

The integral is absolutely convergent, and the statement and proof of Theorem 1.1 applies using the following definition of the functions $f(x)$, $g^\pm_k(x, q)$ and $g(x)$:

$$(57a) \quad f(x - c_b) = \Phi_b(x)^2 \Phi_b(2x - c_b) e^{-2\pi i x^2}$$

$$(57b) \quad g^\pm_k(x, q) = \frac{q^{k(k+1)} x^{2k}}{(qx; q)_k^2 (qx^2; q)_{2k}}$$

$$(57c) \quad g(x) = \frac{x^2}{(1-x)^2 (1-x^2)^2}.$$

Observe that $f(z)$ is non-vanishing and absolutely integrable along the line \mathcal{C}_λ if

$$(58) \quad -(M + N)/4 < \lambda < 0,$$

and $f(z + \mathbf{si}) \neq f(z)$ if λ is in general position.

We now discuss the solutions of the gluing equations $g(x) = 1$ and the matching with the set of nonabelian parabolic $\mathrm{PSL}(2, \mathbb{C})$ representations, illustrating Remark 1.6.

The equation $g(x) = 1$ has 6 solutions that come from two cubic equations:

$$(59) \quad \frac{z}{(1-z^2)(1-z)} = \pm 1.$$

Each triple of solutions lies in number fields F_+ and F_- of discriminant -23 and 49 and type $[1, 1]$ and $[3, 0]$ respectively.

On the other hand, there are 6 nonabelian parabolic $\mathrm{PSL}(2, \mathbb{C})$ representations of the $(-2, 3, 7)$ pretzel knot. These may be found using the Ptolemy methods of [GGZ14] and their **snappy** implementation [CDW]. An alternative method is to use the A -polynomial of the pretzel knot from [Cul]

$$A(m, l) = l^6 - l^5 m^8 + 2l^5 m^9 - l^5 m^{10} - 2l^4 m^{18} - l^4 m^{19} + l^2 m^{36} + 2l^2 m^{37} + l m^{45} - 2l m^{46} + l m^{47} - m^{55}$$

Observe that $A(1, l) = (l-1)^3(l+1)^3$. Setting $(m, l) = (1+t, \pm 1 + c_\pm t + O(t^2))$ we obtain that

$$-6119 + 2012c_- - 220c_-^2 + 8c_-^3 = 0, \quad -6193 - 2020c_+ - 220c_+^2 - 8c_+^3 = 0$$

Then, we have $F_\pm = \mathbb{Q}(c_\pm)$. If z is a solution to (59), let ρ_z denote the corresponding nonabelian parabolic $\mathrm{PSL}(2, \mathbb{C})$ representation. The Rogers dilogarithm of z agrees with the complex volume of ρ_z , and $g'(x)$ agrees with the 1-loop invariant of ρ_z .

Incidentally, if $z \in F_+$, a totally real field, then the corresponding triple of elements of the Bloch group is torsion and triple of complex volumes is given by

$$\left(e\left(-\frac{19}{42}\right), e\left(-\frac{13}{42}\right), e\left(\frac{11}{42}\right) \right) = e\left(-\frac{19}{42}\right) \left(1, e\left(\frac{1}{7}\right), e\left(-\frac{2}{7}\right) \right),$$

where $e(x)$ is given by Equation (20).

3. PROOF OF THEOREM 1.9

We start by taking the logarithmic derivative of Faddeev's quantum dilogarithm

$$\begin{aligned}
 (60) \quad \frac{\partial}{\partial x} \log \Phi_{\mathbf{b}}(x) &= \int_{\mathbb{R}+i\epsilon} \frac{-2ie^{-2ixz}}{4 \sinh(z\mathbf{b}) \sinh(z\mathbf{b}^{-1})} dz \\
 &= \int_{\mathbb{R}+i\epsilon} \frac{e^{-2ixz}}{2i \sinh(z\mathbf{b}) \sinh(z\mathbf{b}^{-1})} dz \\
 &= \int_{\mathbb{R}+i\epsilon} \frac{\pi \mathbf{s} e^{-2\pi i \mathbf{s} x z}}{2i \sinh(\pi z \mathbf{b} \mathbf{s}) \sinh(\pi z \mathbf{b}^{-1} \mathbf{s})} dz \\
 &= \int_{\mathbb{R}+i\epsilon} \frac{\pi \mathbf{s} e^{-2\pi i \mathbf{s} x z}}{2i \sinh(\pi z M) \sinh(\pi z N)} dz.
 \end{aligned}$$

After rescaling $x \mapsto \frac{x}{2\pi \mathbf{s}}$ we obtain

$$(61) \quad 4i \frac{\partial}{\partial x} \log \Phi_{\mathbf{b}} \left(\frac{x}{2\pi \mathbf{s}} \right) = \int_{\mathbb{R}+i\epsilon} \frac{e^{-ixz}}{\sinh(\pi z M) \sinh(\pi z N)} dz.$$

The integrand in (61), given by the function

$$(62) \quad f(z) = \frac{e^{-ixz}}{\sinh(\pi z M) \sinh(\pi z N)}$$

satisfies Equation (26) with $a = i$ as a direct consequence of the equalities

$$(63) \quad \frac{f(z \pm i)}{f(z)} = (-1)^{M+N} e^{\pm x}.$$

Equation (27) and an application of Cauchy's residue theorem implies that

$$\begin{aligned}
 (64) \quad \frac{2}{\pi} (1 - (-1)^{M+N} e^x) \frac{\partial}{\partial x} \log \Phi_{\mathbf{b}} \left(\frac{x}{2\pi \mathbf{s}} \right) &= \frac{1}{2\pi i} \left(\int_{\mathbb{R}+i\epsilon} - \int_{\mathbb{R}+i(1+\epsilon)} \right) f(z) dz \\
 &= S_1(z) + S_2(z) + S_3(z),
 \end{aligned}$$

where

$$S_1 = \sum_{m=1}^{M-1} \text{Res}_{z=i\frac{m}{M}} f(z), \quad S_2 = \sum_{n=1}^{N-1} \text{Res}_{z=i\frac{n}{N}} f(z), \quad S_3 = \text{Res}_{z=i} f(z).$$

So, we have reduced the integrals to the sum of residues. Our next task is to calculate each residue. Let us introduce C_i for $i = 1, 2, 3$ by:

$$\begin{aligned}
 C_1 &= \frac{(1 - e^{x+\pi i(M+N)})^{\frac{M-1}{2M}}}{D_M(e^{(x+\pi i(M+N))/M}; e^{2\pi i N/M})} \\
 C_2 &= \frac{(1 - e^{x+\pi i(M+N)})^{\frac{N-1}{2N}}}{D_N(e^{(x+\pi i(M+N))/N}; e^{2\pi i M/N})} \\
 C_3 &= (1 - (-1)^{M+N} e^x)^{\frac{ix}{2\pi \mathbf{s}^2}} e^{\frac{i}{2\pi \mathbf{s}^2}} \text{Li}_2((-1)^{M+N} e^x).
 \end{aligned}$$

Lemma 3.1. For $i = 1, 2, 3$ we have:

$$(65) \quad S_i = \frac{2}{\pi} (1 - (-1)^{M+N} e^x) \frac{\partial}{\partial x} \log C_i.$$

Proof. First we compute S_1 . Expanding in powers of z around $z = 0$, we have

$$(66) \quad f\left(z + i\frac{m}{M}\right) = \frac{(-1)^m e^{mx/M}}{\pi z M i \sin(\pi m N/M)} (1 + O(z)) = \frac{(-1)^m e^{mx/M}}{\pi z M i \sin(\pi m N/M)} + O(1)$$

so that

$$\begin{aligned} \operatorname{Res}_{z=i\frac{m}{M}} f(z) &= \frac{(-e^{x/M})^m}{\pi i M \sin(\pi m N/M)} = \frac{2(-e^{x/M})^m}{\pi M (e^{\pi i m N/M} - e^{-\pi i m N/M})} \\ &= \frac{-2(-e^{(x+\pi i N)/M})^m}{\pi M (1 - e^{2\pi i m N/M})} = \frac{-2e^{m(x+\pi i(M+N))/M}}{\pi M (1 - e^{2\pi i m N/M})}. \end{aligned}$$

Now, by using Lemma 3.2 (see below), we calculate

$$\begin{aligned} -\frac{\pi}{2} \sum_{m=1}^{M-1} \operatorname{Res}_{z=i\frac{m}{M}} f(z) &= M^{-1} \sum_{m=1}^{M-1} \frac{e^{m(x+\pi i(M+N))/M}}{1 - e^{2\pi i m N/M}} \\ &= \frac{M-1}{2M} e^{x+\pi i(M+N)} + (1 - e^{x+\pi i(M+N)}) \frac{\partial}{\partial x} \log D_M(e^{(x+\pi i(M+N))/M}; e^{2\pi i N/M}). \end{aligned}$$

Finally observe that

$$-\int_{-\infty}^x \frac{e^{y+\pi i(M+N)}}{(1 - e^{y+\pi i(M+N)})} dy = \log(1 - e^{x+\pi i(M+N)})$$

This proves Equation (65) for $i = 1$. Interchanging M with N proves Equation (65) for $i = 2$. Finally we compute S_3 . Expanding in powers of z around $z = 0$, we have

$$\begin{aligned} (-1)^{M+N} e^{-x} f(z + i) &= f(z) = \frac{1 - ixz + O(z^2)}{\pi z M (1 + O(z^2)) \pi z N (1 + O(z^2))} \\ &= \frac{1 - ixz + O(z^2)}{\pi^2 M N z^2} = \frac{1}{\pi^2 S^2 z^2} - \frac{ix}{\pi^2 S^2 z} + O(1) \end{aligned}$$

so that

$$\operatorname{Res}_{z=i} f(z) = \frac{(-1)^{1+M+N} i x e^x}{\pi^2 S^2}.$$

Now we calculate

$$\begin{aligned} 2\pi i S^2 \log C_3 &= \int_{-\infty}^x \frac{(-1)^{M+N} y e^y dy}{1 - (-1)^{M+N} e^y} = - \int_{-\infty}^x y d \log(1 - (-1)^{M+N} e^y) \\ &= - [y \log(1 - (-1)^{M+N} e^y)]_{-\infty}^x + \int_{-\infty}^x \log(1 - (-1)^{M+N} e^y) dy \\ &= -x \log(1 - (-1)^{M+N} e^x) + \int_0^{(-1)^{M+N} e^x} \frac{\log(1 - z)}{z} dz \\ &= -x \log(1 - (-1)^{M+N} e^x) - \operatorname{Li}_2((-1)^{M+N} e^x). \end{aligned}$$

Equation (65) follows for $i = 3$. □

We now finish the proof of Theorem 1.9. Using

$$(67) \quad \lim_{x \rightarrow -\infty} \Phi_b(x) = 1$$

it follows that

$$(68) \quad \log \Phi_b\left(\frac{x}{2\pi s}\right) = \int_{-\infty}^x \frac{\partial}{\partial y} \log \Phi_b\left(\frac{y}{2\pi s}\right) dy$$

Combining the above with Equation (64) and Lemma 3.1, we obtain that

$$(69) \quad \Phi_b\left(\frac{x}{2\pi s}\right) = C_1 C_2 C_3$$

Introduce a new variable z related to x by

$$(70) \quad \frac{x}{2\pi s} = \frac{z}{2\pi s} - c_b.$$

In other words, we have

$$(71) \quad x = z - \pi i(M + N).$$

Equation (68) implies that

$$(72) \quad \Phi_b\left(\frac{z}{2\pi s} - c_b\right) D_N(e^{z/N}; q_+) D_M(e^{z/M}; q_-) e^{-\frac{i}{2\pi s^2} \text{Li}_2(e^z)} \\ = (1 - e^z)^{\frac{M-1}{2M} + \frac{N-1}{2N} + \frac{i(z - \pi i(M+N))}{2\pi MN}} = (1 - e^z)^{1 + \frac{iz}{2\pi MN}}.$$

This concludes the proof of Theorem 1.9. \square

Lemma 3.2. For any complex root of unity q of order M , we have

$$(73) \quad \sum_{m=1}^{M-1} \frac{x^m}{1 - q^m} = \frac{M-1}{2} x^M + (1 - x^M) x \frac{\partial}{\partial x} \log D_M(x; q).$$

Proof. We calculate

$$(1 - x^M) x \frac{\partial}{\partial x} \log D_M(x; q) = (1 - x^M) x \frac{\partial}{\partial x} \sum_{m=1}^{M-1} \frac{m}{M} \log(1 - xq^m) \\ = -\frac{(1 - x^M)x}{M} \sum_{m=1}^{M-1} \frac{mq^m}{1 - xq^m} = -\frac{x}{M} \sum_{m=1}^{M-1} \frac{mq^m (1 - (xq^m)^M)}{1 - xq^m} \\ = -\frac{x}{M} \sum_{m=1}^{M-1} mq^m \sum_{n=0}^{M-1} (xq^m)^n = -\frac{1}{M} \sum_{n=0}^{M-1} x^{n+1} \sum_{m=1}^{M-1} mq^{m(n+1)} \\ = -\frac{1}{M} \sum_{n=1}^M x^n \sum_{m=1}^{M-1} mq^{mn} = -\frac{M-1}{2} x^M - \frac{1}{M} \sum_{n=1}^{M-1} x^n \sum_{m=1}^{M-1} mq^{mn}.$$

To finish the proof, we do the final calculation

$$(74) \quad \sum_{m=1}^{M-1} mq^{mn} = t \frac{\partial}{\partial t} \sum_{m=1}^{M-1} t^m \Big|_{t=q^n} = t \frac{\partial}{\partial t} \left(\frac{1 - t^M}{1 - t} - 1 \right) \Big|_{t=q^n} = \frac{-Mt^M}{1 - t} \Big|_{t=q^n} = \frac{-M}{1 - q^n}.$$

□

APPENDIX A. SOME USEFUL PROPERTIES OF THE QUANTUM DILOGARITHM

The quantum dilogarithm $\Phi_{\mathbf{b}}(x)$ is defined by [Fad95]

$$(75) \quad \Phi_{\mathbf{b}}(x) = \frac{(e^{2\pi\mathbf{b}(x+c_{\mathbf{b}})}; q)_{\infty}}{(e^{2\pi\mathbf{b}^{-1}(x-c_{\mathbf{b}})}; \tilde{q})_{\infty}},$$

where

$$q = e^{2\pi i \mathbf{b}^2}, \quad \tilde{q} = e^{-2\pi i \mathbf{b}^{-2}}, \quad c_{\mathbf{b}} = \frac{i}{2}(\mathbf{b} + \mathbf{b}^{-1}), \quad \text{Im}(\mathbf{b}^2) > 0.$$

An integral representation is given by

$$\Phi_{\mathbf{b}}(x) = \exp \left(\int_{\mathbb{R}+i\epsilon} \frac{e^{-2ixz}}{4 \sinh(z\mathbf{b}) \sinh(z\mathbf{b}^{-1})} \frac{dz}{z} \right)$$

in the strip $|\text{Im}z| < |\text{Im}c_{\mathbf{b}}|$. Remarkably, this function admits an extension to all values of \mathbf{b} with $\mathbf{b}^2 \notin \mathbb{R}_{\leq 0}$. $\Phi_{\mathbf{b}}(x)$ is a meromorphic function of x with

$$\text{poles: } c_{\mathbf{b}} + i\mathbb{N}\mathbf{b} + i\mathbb{N}\mathbf{b}^{-1}, \quad \text{zeros: } -c_{\mathbf{b}} - i\mathbb{N}\mathbf{b} - i\mathbb{N}\mathbf{b}^{-1}.$$

The functional equation

$$\Phi_{\mathbf{b}}(x) \Phi_{\mathbf{b}}(-x) = e^{\pi i x^2} \Phi_{\mathbf{b}}(0)^2, \quad \Phi_{\mathbf{b}}(0) = \left(\frac{q}{\tilde{q}} \right)^{\frac{1}{48}} = e^{\pi i (\mathbf{b}^2 + \mathbf{b}^{-2})/24}$$

allows us to move $\Phi_{\mathbf{b}}(x)$ from the denominator to the numerator of the integrand of a state-integral.

The asymptotics of the quantum dilogarithm are given by [AK14, App.A]

$$(76) \quad \Phi_{\mathbf{b}}(x) \sim \begin{cases} \Phi_{\mathbf{b}}(0)^2 e^{\pi i x^2} & \text{when } \Re(x) \gg 0 \\ 1 & \text{when } \Re(x) \ll 0 \end{cases}$$

The quantum dilogarithm is a quasi-periodic function. Explicitly, it satisfies the equations

$$(77a) \quad \frac{\Phi_{\mathbf{b}}(x + c_{\mathbf{b}} + i\mathbf{b})}{\Phi_{\mathbf{b}}(x + c_{\mathbf{b}})} = \frac{1}{1 - qe^{2\pi\mathbf{b}x}}$$

$$(77b) \quad \frac{\Phi_{\mathbf{b}}(x + c_{\mathbf{b}} + i\mathbf{b}^{-1})}{\Phi_{\mathbf{b}}(x + c_{\mathbf{b}})} = \frac{1}{1 - \tilde{q}^{-1}e^{2\pi\mathbf{b}^{-1}x}}.$$

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